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Sufficient conditions for graphs to be super-λ'

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Abstract. Let $G = (V, E)$ be a connected graph. An edge set $S \subset E$ is a restricted edge cut, if $G - S$ is disconnected and every component of $G - S$ has at least two vertices. The restricted edge connectivity $\lambda'(G)$ of G is the cardinality of a minimum restricted edge cut of G. A graph G is λ'-connected, if restricted edge cuts exist. A graph G is called λ'-optimal, if $\lambda'(G) = \xi(G)$, where $\xi(G) = min\{\xi(e) = d(u) + d(v) - 2 : e = uv \in E\}$. Furthermore, if every minimum restricted edge cut is a set of edges incident to a certain edge, then G is said to be super restricted edge connected or super-λ' for simplicity. Inverse degree of G is $R(G) = \sum_{v\in V} \frac{1}{d(v)}$, where $d(v)$ denotes the degree of the vertex v. We show that let G be a λ'-connected triangle-free graph. If

$$R(G) < 4 - 4\xi(\frac{1}{2\delta(2\delta + 2)} + \frac{1}{(n - 2\delta)(n - 2\delta + 2)}),$$

then G is super-λ'.

Key words. Interconnection networks, Fault-tolerance, Restricted edge connectivity, super-λ', Inverse degree..

1. Introduction

A network is often modeled by a graph $G = (V, E)$ with the vertices representing nodes such as processors or stations, and the edges representing links between the nodes. One fundamental consideration in the design of networks is reliability [2]. An edge cut of a connected graph G is a set of edges whose removal disconnects G. The *edge connectivity* $\lambda(G)$ of G is the minimum cardinality of an edge cut S of G. The edge connectivity $\lambda(G)$ is an important feature determining reliability and fault-tolerance of the network. In the definitions of $\lambda(G)$, no restrictions are imposed on the components of $G - S$. To compensate for this shortcoming, it would

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seem natural to generalize the notion of the classical connectivity by imposing some conditions or restrictions on the components of $G-S$. Following this idea, restricted edge connectivity were proposed in [4,6]

An edge set $S \subset E$ is said to be a *restricted edge cut*, if $G-S$ is disconnected and every component of $G-S$ has at least two vertices. The *restricted edge connectivity* of G, denoted by $\lambda'(G)$, is the cardinality of a minimum restricted-edge-cut of G. If S is a restricted edge cut and $|S| = \lambda'(G)$, then we call S a λ'-*cut*. Esfahanian and Hakimi proved the existence of restricted edge cuts and upper bound for the restricted edge connectivity:

Theorem 1. (*Esfahanian and Hakimi [4]*) *For any connected graph G with at least four vertices which is not isomorphic to the star $K_{1,n-1}$, $\lambda'(G)$ is well defined. Furthermore, $\lambda'(G) \leq \xi(G)$, where $\xi(G) = min\{\xi(e) = d(u) + d(v) - 2 : e = uv \in E\}$ is the minimum edge degree of G.*

G is said to be λ'-*optimal*, if $\lambda'(G) = \xi(G)$. Thus, restricted edge connectivity is generalization of the classical edge connectivity and can provide more accurate measures for the reliability and the fault-tolerance of a large-scale parallel processing system, and so has received much attention in recent years. Let G be a λ'-connected graph, if every minimum restricted edge cut is a set of edges incident to a certain edge, then G is said to be super restricted edge connected or super-λ' for simplicity.

For graph-theoretical terminology and notation not defined here we follow [1]. All graphs considered in this paper are simple, finite and undirected.

Let $G = (V, E)$ be a connected graph, $d_G(v)$ the degree of a vertex v in G (simply $d(v)$), and $\delta(G)$ the minimum degree of G. Moreover, for $S \subset V$, $G[S]$ is the subgraph induced by S. $G-S$ denotes the subgraph of G induced by the vertex set of $V \setminus S$ and $\overline{S} = V - S$. If $u, v \in V$, $d(u, v)$ denotes the length of a shortest (u, v)-path. And the *diameter* is $dm(G) = max\{d(u, v) : u, v \in V\}$. The *girth* of G is the minimum length of cycles in G. For $X, Y \subset V$, denote by $[X, Y]$ the set of edges of G with one end in X and the other in Y.

Define the inverse degree of a graph G with no isolated vertices as

$$R(G) = \sum_{v \in V} \frac{1}{d(v)}.$$

The inverse degree first attracted attention through conjectures of the computer program Graffiti [5]. It has been studied by several authors [3]. In this paper we give sufficient conditions for a triangle-free graph to be super-λ' in terms of $R(G), \delta(G), \xi(G)$ and n.

2. Super-λ' and Inverse Degree

We start the section with the following useful lemmas.

Lemma 1. *Let G be a λ'-connected triangle-free graph. If G is not super-λ', then there exist a λ'-cut $[X, Y]$ with two disjoint sets $X, Y \subset V(G)$, $X \cup Y = V(G)$ and $|[X, Y]| = \lambda'$ such that $|X|, |Y| \geq \xi + 2$.*

Proof. By the hypothesis $|X|, |Y| \geq 3$. Let $X_1 \subseteq X$ be the set of vertices, in which each vertex is incident with at least one edge of $[X, Y]$, and $X_0 = X - X_1$.

Case 1. $X_0 = \varnothing$.

Hence each vertex of X is incident with at least one edge of $[X, Y]$. Take $e = xy \in E(G[X])$, we have

$$\begin{aligned}\xi(G) &\leq d(x) + d(y) - 2,\\ &= |N(x)| + |N(y)| - 2,\\ &= |(N(x) \cap X)\backslash\{y\}| + |N(x) \cap Y| + |(N(y) \cap X)\backslash\{x\}| + |N(y) \cap Y|,\\ &\leq |[\{x, y\}, Y]| + |[X\backslash\{x, y\}, Y]|,\\ &= |[X, Y]| = \lambda'(G).\end{aligned}$$

We get that $\lambda'(G) = \xi(G)$. Since $|X| \geq |N(x) \cup N(y)| = d(x) + d(y)$, we have $\lambda'(G) = |[X, Y]| \geq |X| \geq d(x) + d(y) > \xi(G) = \lambda'(G)$, which is a contradiction.

Case 2. $X_0 \neq \varnothing$.

Subcase 2.1. $G[X_0]$ is an independent set. Let $x \in X_0, y \in X_1$ and $xy \in E(G[X])$, then $N(x) \subseteq X_1$. We can get

$$\begin{aligned}\xi(G) &\leq d(x) + d(y) - 2,\\ &= |N(x)| + |N(y)| - 2,\\ &= |(N(x) \cap X)\backslash\{y\}| + |(N(y) \cap X)\backslash\{x\}| + |N(y) \cap Y|,\\ &\leq |[\{x, y\}, Y]| + |[X\backslash\{x, y\}, Y]|,\\ &= |[X, Y]| = \lambda'(G).\end{aligned}$$

Because of $|X_1| \geq |N(x) \cup N(y) - x| = d(x) + d(y) - 1$, we have $\lambda'(G) = |[X_1, Y]| \geq |X_1| \geq d(x) + d(y) - 1 > \xi(G) = \lambda'(G)$, which is impossible.

Subcase 2.2. Choose an edge $e = xy \in E(G[X_0])$. Since G is a triangle-free graph and $N(x) \cap N(y) = \varnothing$. Hence we can get

$$\begin{aligned}|X| = |X_1| + |X_0| &\geq d(x) - 1 + d(y) - 1 + 2,\\ &\geq d(x) + d(y),\\ &\geq \xi(e) + 2,\\ &\geq \xi(G) + 2.\end{aligned}$$

□

Corollary 1. *Let G be a λ'-connected triangle-free graph of order n. If $n \leq 2\xi(G) + 3$, then G is super-λ'.*

Lemma 2. [3] (1) *Let $a_1, a_2, \cdots, a_p$, A be positive reals with $\sum_{i=1}^{p} a_i \leq A$. Then $\sum_{i=1}^{p}(1/a_i) \geq p^2/A$.*

(2) *If, in addition $a_1, a_2, \cdots, a_p, A$ are positive integers, and a, b are integers with $A = ap + b$ and $0 \leq b \leq p$, then*

$\sum_{i=1}^{p}(1/a_i) \geq (p - b)/a + b/(a + 1)$.

Equality holds if and only if $p-b$ *of the* a_i *equal* a *and the remaining* a_i *equal* $a+1$.

(3) *If* $f(x)$ *is continuous and convex on an interval* $[L,R]$, *and if* $l,r\in[L,R]$, *with* $l+r=L+R$, *then* $f(L)+f(R)\geq f(l)+f(r)$.

Theorem 2. *Let* G *be a* λ'*-connected triangle-free graph of order* n. *If*

$$R(G) < 4-4\xi(\frac{1}{2\delta(2\delta+2)}+\frac{1}{(n-2\delta)(n-2\delta+2)}), \qquad (1)$$

then G *is super-*λ'.

Proof. Let G is not super-λ'. By Lemma 1, then there exist a λ'-cut $[X,Y]$ with two disjoint sets $X,Y\subset V(G)$, $X\cup Y=V(G)$ and $|[X,Y]|=\lambda'$ such that $|X|,|Y|\geq \xi(G)+2\geq 2\delta$, that is $2\delta\leq |X|,|Y|\leq n-2\delta$ and $\delta\leq n/4$. By using Turán's Theorem we have

$$\sum_{v\in X} d(v)\leq 2\lfloor |X|^2/4\rfloor+\lambda'\leq 2\lfloor |X|^2/4\rfloor+\xi. \qquad (*)$$

First let $|X|$ be even. By Lemma 2 (2), $\sum_{v\in X}1/d(v)$ is minimized subject to (*) if ξ degrees equal $|X|/2+1$ and $|X|-\xi$ of the degrees equal $|X|/2$. Hence

$$\sum_{v\in X}\frac{1}{d(v)}\geq \frac{|X|-\xi}{|X|/2}+\frac{\xi}{|X|/2+1}=2-\frac{4\xi}{|X|(|X|+2)}.$$

If $|X|$ is odd, $\sum_{v\in X}1/d(v)$ is minimized subject to (*) if $(|X|-1)/2+\xi$ degrees equal $(|X|+1)/2$ and $(|X|+1)/2-\xi$ of the degrees equal $(|X|-1)/2$. Hence

$$\sum_{v\in X}\frac{1}{d(v)}\geq \frac{(|X|-1)/2+\xi}{(|X|+1)/2}+\frac{(|X|+1)/2-\xi}{(|X|-1)/2}=2-\frac{4(\xi-1)}{(|X|-1)(|X|+1)}.$$

We consider three cases, depending on the parities of n and $|X|$.

Case 1. n is even and X is even.

Then $|Y|=n-|X|$ is also even. By the above inequalities,

$$R(G)=\sum_{v\in V}\frac{1}{d(v)}\geq 4-4\xi(\frac{1}{|X|(|X|+2)}+\frac{1}{(n-|X|)(n-|X|+2)}).$$

Define a function $g(t)=\frac{1}{t(t+2)}$. It is easy to verify that $g''(t)>0$ for $t>0$ and hence the function $g(t)$ is convex. By $2\delta\leq |X|,|Y|\leq n-2\delta$ and Lemma 2 (3), we have $g(|X|)+g(n-|X|)\leq g(2\delta)+g(n-2\delta)$ and thus

$$R(G)=\sum_{v\in V}\frac{1}{d(v)}\geq 4-4\xi(\frac{1}{2\delta(2\delta+2)}+\frac{1}{(n-2\delta)(n-2\delta+2)}).$$

a contradiction.

Case 2. n is even and X is odd.

Then $|Y| = n - |X|$ is also odd and we have $|X|, |Y| \geq 2\delta + 1$. As in Case 1, we have

$$\begin{aligned} R(G) = \sum_{v \in V} \frac{1}{d(v)} &\geq 4 - 4(\xi - 1)(\frac{1}{(|X|-1)(|X|+1)} + \frac{1}{(n-|X|-1)(n-|X|+1)}), \\ &\geq 4 - 4(\xi - 1)(\frac{1}{2\delta(2\delta+2)} + \frac{1}{(n-2\delta)(n-2\delta-2)}). \qquad (2) \end{aligned}$$

A simple calculation can show that (2) > (1) for $\delta \leq n/4$, a contradiction.

Case 3. n is odd.

Without loss of generality, assume that X is odd and $|Y|$ is even. Then $|X| \geq 2\delta + 1$. As above,

$$\begin{aligned} R(G) = \sum_{v \in V} \frac{1}{d(v)} &= \sum_{x \in X} \frac{1}{d(x)} + \sum_{y \in Y} \frac{1}{d(y)}, \\ &\geq 4 - \frac{4(\xi-1)}{(|X|-1)(|X|+1)} - \frac{4\xi}{(n-|X|)(n-|X|+2)}, \\ &\geq 4 - \frac{4(\xi-1)}{2\delta(2\delta+2)} - \frac{4\xi}{(n-2\delta-1)(n-2\delta+1)}. \qquad (3) \end{aligned}$$

A simple calculation can show that (3) > (1) for $\delta \leq n/4$, a contradiction. □

The following example show that the bound is sharp.

Example. For given $n \geq 4\delta, \delta \geq 2$ with n even, let G be the bipartite graph obtained from the disjoint unions of $K_{\delta,\delta}$ with bipartition (A, B) and $K_{n/2-\delta,n/2-\delta}$ with bipartition (C, D), by choosing a set of $\delta - 1$ independent vertices in A and D and adding a matching between the vertices of these sets, and again choosing a set of $\delta - 1$ independent vertices in B and C and adding a matching between the vertices of these sets. Then G is λ'-connected triangle-free and is not super-λ', but

$$\begin{aligned} R(G) &= \frac{2}{\delta} + \frac{2\delta-2}{\delta+1} + \frac{2\delta-2}{n/2-\delta+1} + \frac{n-4\delta+2}{n/2-\delta}, \\ &= 4 - 4(2\delta-2)(\frac{1}{2\delta(2\delta+2)} + \frac{1}{(n-2\delta)(n-2\delta+2)}), \\ &= 4 - 4\xi(\frac{1}{2\delta(2\delta+2)} + \frac{1}{(n-2\delta)(n-2\delta+2)}). \end{aligned}$$

Remark. In the above example, let $\delta-1$ independent vertices in A be $\{x_1, x_2, \cdots, x_{\delta-1}\}$ and $\delta - 1$ independent vertices in D be $\{y_1, y_2, \cdots, y_{\delta-1}\}$. Let $G' = G + x_1y_2$, then $|V(G')| = |V(G)| = n, \delta(G') = \delta(G) = \delta$, and $R(G') < R(G)$. Thus we get that G is super-λ' from Theorem 2.

3. Conclusion

Connectivity is a parameter to measure the reliability of networks. And there are many kinds of connectivities of graphs. Hence we will study the other connectivities of graphs.

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